Geometric Quantization, Complex Structures and the Coherent State Transform

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Abstract

It is shown that the heat operator in the Hall coherent state transform for a compact Lie group K [Ha1] is related with a Hermitian connection associated to a natural one-parameter family of complex structures on T^*K. The unitary parallel transport of this connection establishes the equivalence of (geometric) quantizations of T^*K for different choices of complex structures within the given family. In particular, these results establish a link between coherent state transforms for Lie groups and results of Hitchin [Hi] and Axelrod, Della Pietra and Witten [AdPW].

Contents

1 Introduction

In the present paper we relate the appearance of the heat operator in the Hall coherent state transform (CST) for a compact connected Lie group K [Ha1] with a one-parameter family of complex structures on the cotangent bundle T^*K, in the framework of geometric quantization. The heat equation appears also in the quantization of $\mathbb{R}^{2n}$ and of Chern-Simons theories [AdPW, Hi] and in the related theory of theta functions, where it is associated with the so-called Knizhnik-Zamolodchikov-Bernard-Hitchin (KZBH) connection [Fa, Las, Ra, FMN]. The general case was studied from a cohomological point of view in [Hi].

Our main motivation is to give a differential geometric interpretation to the appearence of the heat equation in the Kähler quantization of T^*K, thus answering a question raised in [Ha3], §1.3. This interpretation is based in the projection of the prequantization connection to the quantum sub-bundle in geometric quantization, as proposed in [AdPW]. The main advantage of this approach consists in the fact that it ensures that the quantum connection is Hermitian. Our method is also complementary to the one of Thiemann [Th1, Th2] where he considers generalized canonical transformations generated by complex valued functions on the phase space. The heat equation appears then naturally as the Schrödinger equation for these complex Hamiltonians or complexifiers.

As shown in [AdPW] §1a, for $\mathbb{R}^{2n}$ the heat equation is associated with independence of the quantization with respect to the choice of a complex structure within the family of complex structures which are invariant under translations.

We consider on T^*K a one-parameter family of complex structures $\{J_s\}_{s\in\mathbb{R}_+}$ induced by the diffeomorphisms

$$\begin{array}{ccccccc} T^*K & \simeq & K\times\mathfrak{K}^* & \simeq & K\times\mathfrak{K} & \stackrel{\psi_s}{\to} & K_{\mathbb{C}} \\ & & & & (x,Y) & \mapsto & xe^{isY}, \end{array} \tag{1.1}$$

where $K_{\mathbb{C}}$ is the complexification of K. Here we identify T^*K with $K\times\mathfrak{K}^*$ by means of left-translation and then with $K\times\mathfrak{K}$ by means of an Ad-invariant inner product $(\cdot\,,\cdot)$ on $\mathfrak{K}=\mathrm{Lie}(K)$.

Together with the canonical symplectic structure ω, the pair (ω, J_s) defines on T^*K a Kähler structure for every $s\in\mathbb{R}_+$. Hall has shown [Ha3] that, when one considers geometric quantization of T^*K, the CST, which has been

proved to be unitary in [Ha1], gives (up to a constant factor) the pairing map between the vertically polarized Hilbert space and the Kähler polarized Hilbert space, provided that one takes into account the half-form correction.

The family of complex structures $\{J_s\}$ is generated by the flow of the vector field $v = \sum_j y^j \frac{\partial}{\partial y^j}$. This vector field is not Hamiltonian but it is given by $v = J_s(\sum_j s y^j X_j)$, where we note that $i \sum_j y^j X_j$ is the Hamiltonian vector field for the complex Hamiltonian $\frac{i}{2}|Y|^2$. This is the complex Hamiltonian used by Thiemann [Th1, Th2] (see also [Ha3]) to generate quantum sates in the holomorphic polarization from the vertically polarized ones. The actions of the vector field v and of the Hamiltonian vector field corresponding to $s\frac{i}{2}|Y|^2$ coincide on J_s-holomorphic functions. This explains the relation between our formalism and the formalism of complexifiers proposed by Thiemann.

In order to associate the heat operator with an Hermitian connection, we collect the prequantum and quantum Hilbert spaces for all $s \in \mathbb{R}_+$ in a prequantum and a quantum Hilbert bundles over $\mathbb{R}_+$,

$$\mathcal{H}^{\mathrm{prQ}} \to \mathbb{R}_+$$

and

$$\mathcal{H}^{\mathrm{Q}} \to \mathbb{R}_+$$

and show that the natural Hermitian connection on $\mathcal{H}^{\mathrm{prQ}}$ induces on $\mathcal{H}^{\mathrm{Q}}$ a connection given by a heat operator (Theorem 1). This connection turns out to be naturally equivalent to the connection obtained by varying $\hbar$ in the CST of Hall (Theorem 4).

Contrary to the flat $\mathbb{R}^{2n}$ case, and its infinite dimensional generalization considered in [AdPW], our family of complex structures is not generated by acting on a fixed one with a family of canonical transformations. It is generated by the flux of a vector field which is not symplectic, but rescales ω. The symplectic structure on T^*K however will be kept fixed throughout the paper.

Notice that, as could be expected from [Ha3], the use of the half-form correction to define the Hermitian structure on $\mathcal{H}^{\mathrm{prQ}}$ plays a decisive role in the appearance of the heat equation for the connection induced on the quantum sub-bundle (see also remark 1).

Our approach to the heat operator and the quantum connection in this setting, can also be related to the Blattner-Kostant-Sternberg (BKS) pairing on the quantum bundle $\mathcal{H}^{\mathrm{Q}}$. This will be the theme of a work in preparation ([FMMN]).

2 The quantum connection and the heat equation

Let K be a compact, connected Lie group. We will consider first the case when K is semisimple and will comment briefly on the case of compact tori, $K = U(1)^n$, at the end of section 2.4.

We start by recalling from [Ha3, Wo] aspects of the geometric prequantization of T^*K but with a natural one-parameter family of complex structures generalizing the fixed complex structure considered by Hall.

The prequantum Hilbert bundle $\mathcal{H}^{\mathrm{prQ}}$ over this family is endowed with a natural Hermitian connection, δ^{prQ}. The quantum connection δ^{Q} induced from δ^{prQ} by orthogonal projection on the quantum Hilbert sub-bundle is then automatically Hermitian. Our main result in the present section is Theorem 1 in which we show that δ^{Q} corresponds in a precise sense to a family of Laplace operators on T^*K.

2.1 Complex structures and the prequantum Hilbert bundle

Consider an Ad-invariant inner product $(\cdot\,,\cdot)$ on $\mathfrak{K} = \mathrm{Lie}(K)$ and $\{X_i\}_{i=1}^n$, $n = \dim K$, a corresponding orthonormal basis for $\mathfrak{K}$ viewed as the space of left-invariant vector fields on K. The canonical 1-form on T^*K is given by $\theta = \sum_{i=1}^n y^i w^i$ where $(y^1, \ldots, y^n)$ are the global coordinates on $\mathfrak{K}$ corresponding to the basis $\{X_i\}_{i=1}^n$, and $\{w^i\}_{i=1}^n$ is the basis of left-invariant 1-forms on K dual to $\{X_i\}_{i=1}^n$, pulled-back to T^*K by the canonical projection. The canonical symplectic 2-form is defined as $\omega = -\mathrm{d}\theta$. We let ϵ denote the Liouville volume form on T^*K, given by

$$\epsilon = \frac{1}{n!}\omega^n \ . \tag{2.1}$$

Following the geometric quantization program we consider the trivial complex line bundle L over T^*K, $L = T^*K \times \mathbb{C}$, with the trivial Hermitian structure. Sections of this bundle are thus just functions on T^*K.

Using the diffeomorphisms ψ_s between T^*K and $K_{\mathbb{C}}$ introduced in (1.1) we produce a family, parameterized by $s \in \mathbb{R}_+$, of complex structures J_s on T^*K by pulling back the canonical complex structure J from $K_{\mathbb{C}}$. Explicitly,

$$J_s = \psi_{s*}^{-1} \circ J \circ \psi_{s*},$$

where ψ_{s*} denotes the push-forward of the map ψ_s.

Proposition 1. *The pair (ω, J_s) defines a Kähler structure on T^*K for every $s \in \mathbb{R}_+$, whose Kähler potential is $\kappa_s(x,Y) = s|Y|^2$.*

Proof. The family of complex structures J_s can also be generated by pulling back a fixed complex structure on T^*K with the family of diffeomorphisms $\varphi_s(x,Y) = (x, sY)$ since $J_s = (\psi_1 \circ \varphi_s)_*^{-1} \circ J \circ (\psi_1 \circ \varphi_s)_* = \varphi_{s*}^{-1} \circ J_1 \circ \varphi_{s*}$. Therefore (ω, J_s) defines a Kähler structure on T^*K for every $s \in \mathbb{R}_+$ if and only if $((\varphi_s^{-1})^*\omega = \omega/s, J_1)$ defines a Kähler structure on T^*K. This follows from the fact that (T^*K, ω, J_1) is a Kähler manifold as shown in [Ha3]. In this reference, the Kähler potential of (T^*K, ω, J_1) is computed to be $\kappa(x,Y) = |Y|^2$. Then, the Kähler potential κ_s for (ω, J_s) is $\kappa_s = (\varphi_s^*\kappa)/s = s|Y|^2$. $\square$

Let $\tilde{X}_j$, $j = 1, ..., n$, be the vector fields on T^*K generating the right action of K lifted to T^*K and given by

$$\tilde{X}_j(x,Y) = (X_j, [Y, X_j]). \tag{2.2}$$

Therefore,

$$\psi_{s*}\tilde{X}_j(x,Y) = X_{j,\mathbb{C}}(xe^{isY}),$$

where $X_{j,\mathbb{C}}$ denotes the natural extension of X_j from a left-invariant vector field on $K \subset K_\mathbb{C}$ to the corresponding left-invariant vector field on $K_\mathbb{C}$. Let $\{\tilde{w}^j\}_{j=1}^n$ be the 1-forms defined by $\tilde{w}^j(\tilde{X}_k) = \delta_k^j$ and $\tilde{w}^j(J_s\tilde{X}_k) = 0$, for $j, k = 1, ..., n$. For every $s \in \mathbb{R}_+$, consider also the frame of J_s-holomorphic 1-forms given by

$$\left\{\tilde{\eta}_s^j = \tilde{w}^j - iJ_s\tilde{w}^j\right\}_{j=1}^n ,$$

where $(J_s w)(X) = w(J_s X)$, for a vector field X and a 1-form w on T^*K. Consider also the J_s-canonical bundle on T^*K whose sections are J_s-holomorphic n-forms with natural Hermitian structure defined as follows. For a J_s-holomorphic n-form α_s, let $|\alpha_s|$ be the unique non-negative C^∞ function on T^*K such that $\overline{\alpha}_s \wedge \alpha_s = |\alpha_s|^2 b\epsilon$, where $b = (2i)^n(-1)^{n(n-1)/2}$. Following [Ha3] we write

$$|\alpha_s|^2 = \frac{\overline{\alpha}_s \wedge \alpha_s}{b\epsilon}.$$

A global nowhere vanishing (trivializing) J_s-holomorphic section of the J_s-canonical bundle is given by

$$\Omega_s \equiv \tilde{\eta}_s^1 \wedge \cdots \wedge \tilde{\eta}_s^n.$$

Let us introduce the bundle of half-forms. Let δ_s denote a square root of the J_s-canonical bundle with a fixed trivializing section whose square is Ω_s. As in [Ha3] we denote this section by $\sqrt{\Omega_s}$. Smooth sections of $L \otimes \delta_s$ are of the form

$$\sigma_s = f\sqrt{\Omega_s}, \quad f \in C^\infty(T^*K).$$

The Hermitian structure on the line bundle $L \otimes \delta_s$ is given by

$$\langle \sigma_s, \tilde{\sigma}_s \rangle = \bar{f}\tilde{f}\left(\frac{\overline{\Omega}_s \wedge \Omega_s}{b\epsilon}\right)^{\frac{1}{2}} = \bar{f}\tilde{f}|\Omega_s|. \tag{2.3}$$

Definition 1. The *prequantum bundle* $\mathcal{H}^{\mathrm{prQ}} \to \mathbb{R}_+$ is the Hilbert vector bundle with fiber over $s \in \mathbb{R}_+$ given by

$$\mathcal{H}^{\mathrm{prQ}}_s = \overline{\mathcal{V}^{\mathrm{prQ}}_s}$$

where the bar denotes norm completion, $\mathcal{V}^{\mathrm{prQ}}_s$ is

$$\mathcal{V}^{\mathrm{prQ}}_s = \left\{\sigma_s \in \Gamma^\infty(L \otimes \delta_s) : \, ||\sigma_s||^{\mathrm{prQ}}_s < \infty\right\},$$

$\Gamma^\infty(L \otimes \delta_s)$ denotes the space of C^∞ sections of the bundle $L \otimes \delta_s$, and

$$\langle \sigma_s, \sigma_s \rangle^{\mathrm{prQ}}_s = \int_{T^*K} |\sigma_s|^2 \epsilon.$$

From (2.3) it is easy to see that sections of $\mathcal{H}^{\mathrm{prQ}}$ of the form

$$\frac{f}{\sqrt{|\Omega_s|}}\sqrt{\Omega_s}, \quad f \in L^2(T^*K, \epsilon) \tag{2.4}$$

have s-independent norm.

We choose the smooth Hilbert bundle structure on $\mathcal{H}^{\mathrm{prQ}}$ as the one compatible with the global trivializing map

$$\begin{aligned} \mathbb{R}_+ \times L^2(T^*K, \epsilon) &\to \mathcal{H}^{\mathrm{prQ}} && (2.5)\\ (s, f) &\mapsto \frac{f}{\sqrt{|\Omega_s|}}\sqrt{\Omega_s} && (2.6) \end{aligned}$$

2.2 The prequantum connection – δ^{prQ}

We now introduce a natural Hermitian connection on $\mathcal{H}^{\mathrm{prQ}}$. Before giving its precise definition we state the following proposition, which is a straightforward consequence of [Ha2, Ha3]. Let $\eta(Y)$ be the Ad_K-invariant function defined for Y in a Cartan subalgebra by the following product over the set R^+ of positive roots of $\mathfrak{K}$,

$$\eta(Y) = \prod_{\alpha\in R^+} \frac{\sinh \alpha(Y)}{\alpha(Y)}. \tag{2.7}$$

Let dg be the Haar measure on $K_{\mathbb{C}}$. We then have,

Proposition 2. *The following identities hold:*

1. $|\Omega_s| \equiv \sqrt{\frac{\overline{\Omega}_s\wedge\Omega_s}{b\epsilon}} = s^{\frac{n}{2}}\eta(sY)$*;*
2. $dg_s := (\psi_s)^*(dg) = s^n\eta^2(sY)\epsilon = |\Omega_s|^2\epsilon$*,*

where η is the function on T^*K defined by equation (2.7).

Proof. In [Ha2, Ha3] it is shown that

$$b\,\eta^2(Y)\epsilon = \overline{\Omega}_1\wedge\Omega_1.$$

This is exactly the first identity with $s=1$. Recall from Proposition 1 that $J_s = \varphi_{s*}^{-1}\circ J_1\circ\varphi_{s*}$. This implies the equality $\varphi_s^*(J_1\beta) = J_s(\varphi_s^*\beta)$ for all 1-forms β on T^*K. Therefore,

$$\varphi_s^*(\tilde{\eta}_1^i) = \varphi_s^*(\tilde{w}^i) - i\varphi_s^*(J_1\tilde{w}^i) = \tilde{w}^i - iJ_s\tilde{w}^i = \tilde{\eta}_s^i.$$

Moreover $\varphi_s^*\epsilon = s^n\epsilon$ and this proves the first equation. For the second identity, let

$$\left\{\eta_{\mathbb{C}}^i = w_{\mathbb{C}}^i - iJw_{\mathbb{C}}^i\right\}_{i=1}^n$$

be a basis of left $K_{\mathbb{C}}$-invariant J-holomorphic 1-forms on $K_{\mathbb{C}}$, where $w_{\mathbb{C}}^i$ is the natural extension of w^i from K to $K_{\mathbb{C}}$, obtained by left translations. Then, the Haar measure on $K_{\mathbb{C}}$ is given by

$$dg = \frac{1}{b}\,\overline{\Omega}_{\mathbb{C}}\wedge\Omega_{\mathbb{C}},$$

where $\Omega_{\mathbb{C}} = \eta^1_{\mathbb{C}} \wedge \cdots \wedge \eta^n_{\mathbb{C}}$. Since $\psi_s^* \circ J = J_s \circ \psi_s^*$ for all 1-forms on $K_{\mathbb{C}}$, and $\tilde{w}^i = \psi_s^*(w^i_{\mathbb{C}})$, we have

$$\psi_s^*(\eta^i_{\mathbb{C}}) = \psi_s^*(w^i_{\mathbb{C}}) - i\psi_s^*(Jw^i_{\mathbb{C}}) = \tilde{w}^i - iJ_s\tilde{w}^i = \tilde{\eta}^i_s.$$

Using the previous result we get the desired identity. □

The trivializing section $\sqrt{\Omega_s}$ of the half-form bundle δ_s is canonical (up to a sign) because it is obtained from geometric quantization data (the one parameter family of Kähler polarizations J_s on T^*K) with the only additional structure provided by a fixed Ad-invariant inner product in the Lie algebra $\mathfrak{K}$. This motivates the definition of the prequantum connection as the connection induced from the canonical connection on the trivial bundle by the trivialization of $\mathcal{H}^{\mathrm{prQ}}$ given in (2.5).

Definition 2. The *prequantum connection* δ^{prQ} on $\mathcal{H}^{\mathrm{prQ}}$ is the connection for which sections of the form (2.4) are horizontal

$$\delta^{\mathrm{prQ}} \left(\frac{f}{\sqrt{|\Omega_s|}} \sqrt{\Omega_s} \right) = 0, \tag{2.8}$$

for all $f \in L^2(T^*K, \epsilon)$.

One can also obtain the same connection using a BKS-type pairing on the prequantum bundle $\mathcal{H}^{\mathrm{prQ}}$. This will be explored in [FMMN]. Note that the prequantum connection is Hermitian, that is, it is compatible with the Hermitian structure on $\mathcal{H}^{\mathrm{prQ}}$ in the following sense

$$\frac{d}{ds} \langle \sigma, \zeta \rangle^{\mathrm{prQ}} = \langle \delta^{\mathrm{prQ}}_{\frac{\partial}{\partial s}} \sigma, \zeta \rangle^{\mathrm{prQ}} + \langle \sigma, \delta^{\mathrm{prQ}}_{\frac{\partial}{\partial s}} \zeta \rangle^{\mathrm{prQ}} \tag{2.9}$$

for all smooth sections σ, ζ of $\mathcal{H}^{\mathrm{prQ}}$.

2.3 The induced quantum connection – δ^{Q}

The Kähler polarizations (ω, J_s) enter already the definition of the prequantum Hilbert spaces $\mathcal{H}^{\mathrm{prQ}}_s$ through the half-form bundles δ_s and the Hermitian structures (2.3). To define the fibers $\mathcal{H}^{\mathrm{Q}}_s$ of the quantum Hilbert sub-bundle $\mathcal{H}^{\mathrm{Q}} \subset \mathcal{H}^{\mathrm{prQ}}$, one considers polarized, or J_s-holomorphic, sections of $L \otimes \delta_s$.

Explicitly, for every $s \in \mathbb{R}_+$, consider the frame of left K-invariant vector fields on T^*K
$$\left\{ Z_{j,s} = \frac{1}{2}(X_j - iJ_s X_j) \right\}_{j=1}^{n} .$$
Let the polarizations be given, for every $s \in \mathbb{R}_+$, by

$$\mathcal{P}^s_{(x,Y)} = \mathrm{span}_{\mathbb{C}} \left\{ \bar{Z}_{j,s}(x, Y) \right\}_{j=1}^{n},$$

where $\bar{Z}_{j,s} = \frac{1}{2}(X_j + iJ_s X_j)$. We use the notation $\bar{Z} \in \mathcal{P}^s$ for $\bar{Z}_{(x,Y)} \in \mathcal{P}^s_{(x,Y)}$ for all $(x, Y) \in T^*K$. Note that these polarizations $\{\mathcal{P}^s\}_{s\in\mathbb{R}_+}$ converge, as s tends to zero, to the vertical polarization of T^*K, spanned at every point by $\{\frac{\partial}{\partial y^i}\}_{i=1}^n$.

Definition 3. The *quantum bundle* $\mathcal{H}^{\mathrm{Q}} \to \mathbb{R}_+$ is the Hilbert sub-bundle of $\mathcal{H}^{\mathrm{prQ}}$ with fiber over $s > 0$ given by

$$\mathcal{H}^{\mathrm{Q}}_s = \overline{\mathcal{V}^{\mathrm{Q}}_s}$$

where
$$\mathcal{V}^{\mathrm{Q}}_s = \left\{ f\sqrt{\Omega_s} \in \mathcal{V}^{\mathrm{prQ}}_s : \ \nabla_{\bar{Z}} f = 0, \ \ \forall \ \bar{Z} \in \mathcal{P}^s \right\},$$
and $\nabla_{\bar{Z}} = \bar{Z} - \frac{1}{i\hbar_0}\theta(\bar{Z})$ is the geometric quantization connection defined on the trivial bundle L and $\hbar_0$ is Planck's constant. We call the solutions of $\nabla_{\bar{Z}} f = 0$ the *polarized sections of* L.

Proposition 3. *For every $s > 0$, the $\mathcal{P}^s$-polarized (or J_s-holomorphic) sections of L are the C^∞ functions f on T^*K of the form*

$$f = F\, e^{-\frac{s|Y|^2}{2\hbar_0}}$$

where F is an arbitrary J_s-holomorphic function on T^*K.

Proof. From [Ha3, Proposition 2.3] the solutions of $\nabla_{\bar{Z}_{j,s}} f = 0$ are

$$f = F\, e^{-\kappa_s/2\hbar_0},$$

where F is a J_s-holomorphic function on T^*K, so that the result follows from proposition 1. □

Let us denote by $\langle \cdot, \cdot \rangle^{\mathrm{Q}}$ the Hermitian structure on $\mathcal{H}^{\mathrm{Q}}$ inherited from $\mathcal{H}^{\mathrm{prQ}}$. We conclude that the fibers $\mathcal{H}^{\mathrm{Q}}_s$ of $\mathcal{H}^{\mathrm{Q}}$ are given by

$$\mathcal{H}^{\mathrm{Q}}_s = \left\{ \sigma_s = F e^{-\frac{s|Y|^2}{2\hbar_0}} \sqrt{\Omega_s},\ F \text{ is } J_s\text{-holomorphic and } ||\sigma_s||^Q_s < \infty \right\} .$$

The quantum Hilbert bundle inherits from $(\mathcal{H}^{\mathrm{prQ}}, \delta^{\mathrm{prQ}})$ a Hermitian connection δ^{Q} which we call the *quantum connection*. The parallel transport with respect to this connection is automatically unitary and it establishes the invariance of the quantization of T^*K with respect to the choice of polarization within the family $\{\mathcal{P}^s\}_{s\in\mathbb{R}_+}$.

Definition 4. The *quantum connection* δ^{Q} is the Hermitian connection induced on $\mathcal{H}^{\mathrm{Q}}$ by the natural connection δ^{prQ} on $\mathcal{H}^{\mathrm{prQ}}$

$$\delta^{\mathrm{Q}} = P \circ \delta^{\mathrm{prQ}} ,$$

where P denotes the orthogonal projection $\mathcal{H}^{\mathrm{prQ}} \to \mathcal{H}^{\mathrm{Q}}$.

Below in theorem 1 we will obtain an explicit expression for δ^{Q}. Consider the second order differential operator $\Delta^s_{\mathbb{C}}$ on T^*K given by the pull-back, with respect to ψ_s, of the second order Casimir operator on $K_{\mathbb{C}}$,

$$\Delta^s_{\mathbb{C}} = \sum_{i=1}^{n} (\tilde{X}_i)^2 - (J_s\tilde{X}_i)^2.$$

Note that $\Delta^s_{\mathbb{C}}$ takes J_s-holomorphic functions to J_s-holomorphic functions.

Let $\widehat{\Delta}^s_{\mathbb{C}}$ be the (unbounded) operator on $\mathcal{H}^{\mathrm{Q}}_s$ defined on its dense domain by

$$\widehat{\Delta}^s_{\mathbb{C}} \left(F e^{-s|Y|^2/\hbar_0} \sqrt{\Omega_s} \right) = \Delta^s_{\mathbb{C}} \left[F \right] e^{-s|Y|^2/\hbar_0} \sqrt{\Omega_s}. \tag{2.10}$$

Theorem 1. *Let F be the function on $\mathbb{R}_+ \times T^*K$ obtained, for every $s \in \mathbb{R}_+$, as the pull-back of a given J-holomorphic function $\hat{F}$ on $K_{\mathbb{C}}$,*

$$F(s, x, Y) = \hat{F}(xe^{isY}), \tag{2.11}$$

such that $F(s, \cdot)e^{-\frac{s|Y|^2}{2\hbar_0}} \sqrt{\Omega_s} \in \mathcal{H}^{\mathrm{Q}}_s$ is in the domain of the operator $\widehat{\Delta}^s_{\mathbb{C}}$. The quantum connection δ^{Q} acts on sections of $\mathcal{H}^{\mathrm{Q}}$ of the form

$$F\, e^{-\frac{s|Y|^2}{2\hbar_0}} \sqrt{\Omega_s} \tag{2.12}$$

as

$$\delta^{\mathrm{Q}}_{\frac{\partial}{\partial s}}[F\, e^{-\frac{s|Y|^2}{2\hbar_0}}\sqrt{\Omega_s}] = \frac{\hbar_0}{2}\left(-\frac{1}{2}\Delta^s_{\mathbb{C}} + |\rho|^2\right)[F]\; e^{-\frac{s|Y|^2}{2\hbar_0}}\sqrt{\Omega_s}, \tag{2.13}$$

where ρ is half the sum of the positive roots of $\mathfrak{K}$.

Notice that choosing the sections of $\mathcal{H}^{\mathrm{Q}}$ in the form (2.11) and (2.12) corresponds to choosing moving frames, more precisely a class of global moving frames related by s-independent transformations. Having made such a choice the covariant derivative in the direction of $\partial/\partial s$ is defined by a linear operator acting on the fibers as in (2.13).

We will divide the proof of this theorem in several lemmata. Let us denote by W the following vector field on T^*K,

$$W = i\sum_{j=1}^{n} y^j Z_{j,s}\ .$$

Note that, from (2.2),

$$W = \frac{i}{2}\sum_{j=1}^{n} y^j (X_j - iJ_s X_j) = \frac{i}{2}\sum_{j=1}^{n} y^j(\tilde{X}_j - iJ_s\tilde{X}_j).$$

Lemma 1. *Let $\hat{F}$ be a fixed C^∞ function on $K_{\mathbb{C}}$, F be the function on $\mathbb{R}_+ \times T^*K$, given by $F(s,\cdot) = \psi_s^*\hat{F}$, and let $f \in C^\infty(T^*K)$ be a function only of Y. We have,*

i) If $\hat{F}$ is holomorphic then

$$\frac{\partial F}{\partial s} = WF$$

ii) If $\hat{F}$ is right K-invariant then

$$\frac{\partial F}{\partial s} = 2\ WF = 2\ \bar{W}F$$

iii)

$$Wf = \frac{1}{2s}\sum_{j=1}^{n} y^j \frac{\partial}{\partial y^j}\ f.$$

Proof. A direct computation gives,

$$\frac{\partial}{\partial s}(\psi_s^* \hat{F})(x,Y) = \frac{\partial}{\partial s}\hat{F}(xe^{isY}) = \sum_{j=1}^{n} y^j J_s \tilde{X}_j F.$$

The special cases *i)* and *ii)* above follow from this equation.

The identity in *iii)* follows from

$$\sum_{j=1}^{n} y^j J_s X_j = \frac{1}{s}\sum_{j=1}^{n} y^j \frac{\partial}{\partial y^j}.$$

$\square$

Lemma 2. *Let X be a smooth vector field on T^*K, with $|X(y^i)| < c\exp(\alpha|Y|)$ for some positive constants c and α, and let $\phi \in C^\infty(T^*K)$ be such that*

$$|\phi(x,Y)| < e^{-\delta|Y|^2},$$

for $|Y| > R$ and fixed positive constants R, δ. Then,

$$\int_{T^*K} \mathcal{L}_X(\phi\epsilon) = 0.$$

Proof. Using Cartan's formula, $\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d$, we have $\mathcal{L}_X(\phi\epsilon) = d(\phi\, \iota_X \epsilon)$, where the symplectic volume form is $\epsilon = w^1 \wedge dy^1 \wedge \cdots \wedge w^n \wedge dy^n$. Since $T^*K \cong K \times \mathfrak{K}$, using Stokes formula we only need to show that

$$\lim_{R\to\infty} \int_{K\times S_R^{n-1}} \phi\, \iota_X \epsilon = 0,$$

where S_R^{n-1} denotes the sphere of radius R in $\mathfrak{K}$ centered at the origin. We have

$$\left| \int_{K\times S_R^{n-1}} \phi\, \iota_X \epsilon \right| < c' e^{-\frac{\delta}{2}R^2},$$

for some positive constant c', which proves the lemma. $\square$

Let $\mathcal{B}$ be the subspace of the space $\mathcal{H}(K_{\mathbb{C}})$ of holomorphic functions on $K_{\mathbb{C}}$, spanned by the holomorphic functions

$$\operatorname{tr}(\pi(g)A), \tag{2.14}$$

where π is an irreducible finite-dimensional representation of K extended to an holomorphic representation of $K_{\mathbb{C}}$ and $A \in \mathrm{End}\, V_\pi$. Let $\mathcal{F}_s$ denote the subspace of $\mathcal{H}_s^{\mathrm{Q}}$ given by sections of the form (2.11) and (2.12) with $\hat{F} \in \mathcal{B}$. It follows from the Lemma 10 of [Ha1] that $\mathcal{F}_s$ is dense in $\mathcal{H}_s^{\mathrm{Q}}$.

Lemma 3. *Let $\sigma, \zeta \in \Gamma(\mathcal{H}^{\mathrm{Q}})$ with $\sigma_s, \zeta_s \in \mathcal{F}_s$, $\forall s \in \mathbb{R}_+$, and*

$$\sigma = F\, e^{-\frac{s|Y|^2}{2\hbar_0}} \sqrt{\Omega_s}, \qquad \zeta = G\, e^{-\frac{s|Y|^2}{2\hbar_0}} \sqrt{\Omega_s}.$$

Then we have,

$$\int_{T^*K} \left(\bar{W} - \frac{|Y|^2}{\hbar_0} + \frac{1}{2}\frac{\partial \ln |\Omega_s|}{\partial s} + \frac{n}{4s} \right) \left[\bar{F}\right] G e^{-\frac{s|Y|^2}{\hbar_0}} |\Omega_s|\, \epsilon = 0.$$

Proof. The vector field W satisfies $\bar{W}(y^j) = \frac{1}{2s} y^j$ and we can apply lemma 2 with $0 < \delta < \frac{s}{\hbar_0}$ to the first term above. Integrating by parts we obtain

$$\begin{aligned} -\int_{T^*K} \bar{W}\left[\bar{F}\right] G e^{-\frac{s|Y|^2}{\hbar_0}} |\Omega_s|\, \epsilon = & \int_{T^*K} \bar{F}\, G\, \bar{W}\, \left[e^{-s|Y|^2/\hbar_0}\right] |\Omega_s|\, \epsilon \\ & + \int_{T^*K} \bar{F}\, G\, e^{-s|Y|^2/\hbar_0}\, \bar{W}\, \left[\ln |\Omega_s|\right] |\Omega_s|\, \epsilon \\ & + \int_{T^*K} \bar{F}\, G\, e^{-s|Y|^2/\hbar_0}\, |\Omega_s|\, \mathcal{L}_{\bar{W}}(\epsilon). \end{aligned} \tag{2.15}$$

For the first term on the r.h.s. of (2.15) we have from *iii)* in lemma 1

$$\bar{W}[e^{-s|Y|^2/\hbar_0}] = \frac{1}{2s}\sum_{j=1}^{n} y^j \frac{\partial}{\partial y^j} e^{-s|Y|^2/\hbar_0} = -\frac{|Y|^2}{\hbar_0} e^{-s|Y|^2/\hbar_0}. \tag{2.16}$$

From *ii)* in lemma 1, we see that

$$\bar{W}[\ln |\Omega_s|] = \frac{1}{2}\frac{\partial \ln |\Omega_s|}{\partial s} - \frac{n}{4s}\ . \tag{2.17}$$

For the third term in (2.15), we have

$$\mathcal{L}_{\bar{W}}(\epsilon) = \frac{n}{n!}\, \mathcal{L}_{\bar{W}}(\omega) \wedge \omega^{n-1}.$$

From $\omega = -d\theta$, Cartan's formula $\mathcal{L}_{\bar{W}} = d \circ \iota_{\bar{W}} + \iota_{\bar{W}} \circ d$ and

$$\iota_{\bar{W}}\, d\theta = \frac{1}{2s}\,\theta + \frac{i}{4}\, d|Y|^2\ ,$$

we obtain $\mathcal{L}_{\bar{W}}(\omega) = (1/2s)\omega$, which implies that

$$\mathcal{L}_{\bar{W}}(\epsilon) = \frac{n}{2s}\,\epsilon. \tag{2.18}$$

Substituting (2.16), (2.17) and (2.18) into (2.15) we obtain the desired result. $\square$

Recall that the prequantum connection is Hermitian, so it satisfies equation (2.9). Consider sections of $\mathcal{H}^{\mathrm{Q}}$,

$$\sigma = F\, e^{-\frac{s|Y|^2}{2\hbar_0}}\sqrt{\Omega_s}, \qquad \zeta = G\, e^{-\frac{s|Y|^2}{2\hbar_0}}\sqrt{\Omega_s}, \tag{2.19}$$

where F, G are as in (2.11). These sections also satisfy (2.9) and we have

$$\langle \delta^{\mathrm{Q}}_{\frac{\partial}{\partial s}}\sigma, \zeta\rangle^{\mathrm{Q}} = \langle \delta^{\mathrm{prQ}}_{\frac{\partial}{\partial s}}\sigma, \zeta\rangle^{\mathrm{prQ}}. \tag{2.20}$$

Lemma 4. *Let σ, ζ be as above. Then, the following identity holds*

$$\langle \delta^{\mathrm{Q}}_{\frac{\partial}{\partial s}}\sigma, \zeta\rangle^{\mathrm{Q}} = s^{-n/2}\int_{K_{\mathbb{C}}} \bar{\hat{F}}\hat{G}\ \hbar_0 \left(\frac{|Y|^2}{2\hbar^2} - \frac{n}{4\hbar}\right)\frac{e^{-|Y|^2/\hbar}}{\eta(Y)}\ dg, \tag{2.21}$$

where dg is the Haar measure on $K_{\mathbb{C}}$, $g = xe^{iY}$ and $\hbar = s\hbar_0$.

Proof. From the definition of horizontal sections (2.8) and from (2.20), we obtain

$$\langle \delta^{\mathrm{Q}}_{\frac{\partial}{\partial s}}\sigma, \zeta\rangle^{\mathrm{Q}} = \int_{T^*K}\overline{\left(\frac{\partial F}{\partial s} - \frac{|Y|^2 F}{2\hbar_0} + \frac{F}{2}\frac{\partial \ln|\Omega_s|}{\partial s}\right)}\, G\, e^{-\frac{s|Y|^2}{\hbar_0}}|\Omega_s|\,\epsilon.$$

We now use lemmata 1 and 3 to simplify the equation above for the quantum connection to get,

$$\langle \delta^{\mathrm{Q}}_{\frac{\partial}{\partial s}}\sigma, \zeta\rangle^{\mathrm{Q}} = \int_{T^*K}\bar{F}G\left(\frac{|Y|^2}{2\hbar_0} - \frac{n}{4s}\right)e^{-s|Y|^2/\hbar_0}|\Omega_s|\ \epsilon,$$

which with the help of ψ_s and proposition 2 gives (2.21). $\square$

Let us introduce the K-averaged heat kernel measure $d\nu_\hbar$ on $K_{\mathbb{C}}$ given by [Ha3]

$$d\nu_\hbar(g) = \nu_\hbar(g)\, dg = c_\hbar\, \frac{e^{-|Y|^2/\hbar}}{\eta(Y)}\, dg, \tag{2.22}$$

and $c_\hbar = (\pi\hbar)^{-n/2} e^{-|\rho|^2\hbar}$, ρ being half the sum of the positive roots. Recall from [Ha1] that $\nu_\hbar$ satisfies the equation

$$\frac{\partial \nu_\hbar}{\partial \hbar} = -\frac{1}{4}\Delta_{\mathbb{C}}\nu_\hbar \tag{2.23}$$

on $K_{\mathbb{C}}$, where

$$\Delta_{\mathbb{C}} = \sum_{i=1}^{n} (X_{i,\mathbb{C}})^2 - (JX_{i,\mathbb{C}})^2 \ ,$$

is the Casimir operator for $K_{\mathbb{C}}$. The equation (2.23) is equivalent to the following equality:

Lemma 5.

$$\left(\frac{|Y|^2}{2\hbar^2} - \frac{n}{4\hbar}\right)\frac{e^{-|Y|^2/\hbar}}{\eta(Y)} = \left(-\frac{1}{8}\Delta_{\mathbb{C}} + \frac{|\rho|^2}{2}\right)\frac{e^{-|Y|^2/\hbar}}{\eta(Y)}.$$

□

We are now ready to prove Theorem 1.

Proof. Consider σ and ζ as in (2.19) and let

$$Z_j = \frac{1}{2}\left(X_{j,\mathbb{C}} - iJX_{j,\mathbb{C}}\right), \qquad j = 1, \cdots n,$$

so that

$$\Delta_{\mathbb{C}} = 2\sum_{j=1}^{n} Z_j^2 + \bar{Z}_j^2.$$

From (2.21) and Lemma 5, since

$$X_{j,\mathbb{C}}\left(\frac{e^{-|Y|^2/\hbar}}{\eta(Y)}\right) = 0, \quad \text{for all } j,$$

we obtain

$$\begin{aligned} \langle \delta^Q_{\frac{\partial}{\partial s}}\sigma, \zeta\rangle^Q &= s^{-n/2}\int_{K_{\mathbb{C}}} \bar{\hat{F}}\hat{G}\ \hbar_0 \left(-\frac{1}{8}\Delta_{\mathbb{C}} + \frac{|\rho|^2}{2}\right)\left[\frac{e^{-|Y|^2/\hbar}}{\eta(Y)}\right] dg = \\ &= s^{-n/2}\int_{K_{\mathbb{C}}} \bar{\hat{F}}\hat{G}\ \frac{\hbar_0}{2}\left(-\sum_{j=1}^{n}\bar{Z}_j^2 + |\rho|^2\right)\left[\frac{e^{-|Y|^2/\hbar}}{\eta(Y)}\right] dg. \end{aligned}$$

Lemma 2 can be applied since $\bar{Z}_j(y^j)$ grows linearly with $|Y|$ (see section 6 in [Ha3]). Using it to integrate twice by parts with respect to $\bar{Z}_j$ and noticing that, from the bi-invariance of dg, $\mathcal{L}_{\bar{Z}_j} dg = 0$ and also $\bar{Z}_j(\hat{G}) = 0$ we obtain the statement of the theorem

$$\langle \delta^Q_{\frac{\partial}{\partial s}} \sigma, \zeta \rangle^Q = \int_{T^*K} \frac{\hbar_0}{2} \overline{(-\frac{1}{2}\Delta^s_{\mathbb{C}} + |\rho|^2)[F]}\, G\, e^{-s|Y|^2/\hbar_0}\, |\Omega_s| \epsilon, \tag{2.24}$$

for sections σ, ζ with values in $\mathcal{F}_s \ni \sigma_s, \zeta_s$, for all $s \in \mathbb{R}_+$. The operator $\widehat{\Delta}^s_{\mathbb{C}}$ in (2.10) is essentially self-adjoint (it has a basis of eigenvectors, with $\hat{F}$'s given by matrix elements of finite dimensional holomorphic representations as in (2.14), with real, and nonpositive, eigenvalues) and the space $\mathcal{F}_s$ is dense in $\mathcal{H}^{\mathrm{Q}}_s$. Therefore, the expression (2.24) implies that $\delta^Q_{\frac{\partial}{\partial s}}$ is given by (2.13) for sections of $\mathcal{H}^{\mathrm{Q}}$ in the form (2.11) and (2.12) and with values in the domain of $\widehat{\Delta}^s_{\mathbb{C}}$, for all $s \in \mathbb{R}_+$. □

Remark 1. *Note that the simple expression (2.21) would not be valid without the inclusion of the half-form correction in the definition of the Hermitian structure on $\mathcal{H}^{\mathrm{prQ}}$ (see (2.3) and proposition 2). The half-form leads to the cancellation of the term proportional to $\partial \ln |\Omega_s| / \partial s$. This is what ultimately leads to the heat operator in the quantum connection.*

2.4 The heat equation

Let Ψ be the diffeomorphism

$$\begin{array}{rcl} \Psi : \mathbb{R}_+ \times T^*K & \to & \mathbb{R}_+ \times K_{\mathbb{C}} \\ (s, (x, Y)) & \mapsto & (s, x e^{isY}), \end{array}$$

so that $\Psi(s, .) = \psi_s$, for all $s \in \mathbb{R}_+$. Consider sections σ of $\mathcal{H}^Q$ of the form

$$\sigma_s = \frac{1}{\sqrt{a_s}} F(s, \cdot)\; e^{-\frac{s|Y|^2}{2\hbar_0}} \sqrt{\Omega_s}, \tag{2.25}$$

where $a_s = (\pi\hbar_0)^{n/2} e^{|\rho|^2 \hbar_0 s}$,

$$F = \Psi^* \widetilde{F} \tag{2.26}$$

and $\widetilde{F}$ is a C^∞ function on $\mathbb{R}_+ \times K_{\mathbb{C}}$, holomorphic in the second variable. We then have,

Theorem 2. *A section of $\mathcal{H}^Q$ of the form (2.25), (2.26) with σ_s in the domain of $\widehat{\Delta}^s_{\mathbb{C}}$, for all $s \in \mathbb{R}_+$, is δ^Q-horizontal if and only if $\widetilde{F}$ is a solution of the following heat equation on $K_{\mathbb{C}}$,*

$$\frac{\partial}{\partial s}\widetilde{F} = \frac{\hbar_0}{4}\Delta_{\mathbb{C}}\widetilde{F}.$$

Proof. Substituting (2.25) in (2.13) we obtain

$$\delta^Q_{\frac{\partial}{\partial s}}\sigma = \Psi^*\left(\frac{\partial}{\partial s}\widetilde{F} - \frac{\hbar_0}{4}\Delta_{\mathbb{C}}\widetilde{F}\right)\frac{e^{-\frac{s|Y|^2}{2\hbar_0}}}{\sqrt{a_s}}\sqrt{\Omega_s}.$$

□

Besides establishing the relation of the heat equation on complex semisimple Lie groups $K_{\mathbb{C}}$ with the geometric quantization of T^*K this result also provides the link with the coherent state transform introduced by Hall in [Ha1].

Concerning the case $K = U(1)^n$, note that the results in proposition 2 and equations (2.22) and (2.23) are still valid if we replace $\eta(sY)$ by 1 and the Weyl vector ρ by 0. Therefore, all the results above apply also to this case.

3 The quantum connection and the CST

The coherent state transform for K defines a parallel transport on a Hilbert bundle over $\mathbb{R}_+$ for an Hermitian connection that we denote by δ^H. In this section we show that this parallel transport is naturally equivalent to the parallel transport of the quantum connection δ^{Q}. This follows from propositions 2 and 3, and from (2.3) and (2.22).

3.1 The CST connection - δ^{H}

Let $\rho_\hbar$, $\hbar > 0$, be the heat kernel for the Laplacian Δ on K associated to the Ad-invariant inner product $(\cdot\,,\cdot)$ on $\mathfrak{K}$, $\Delta = \sum_{i=1}^n X_i^2$. As proved in [Ha1], $\rho_\hbar$ has a unique analytic continuation to $K_{\mathbb{C}}$, also denoted by $\rho_\hbar$. The K-averaged coherent state transform (CST) is defined as the map

$$\begin{aligned} C_\hbar \quad &: \quad L^2(K, dx) \to \mathcal{H}(K_{\mathbb{C}}) \\ (C_\hbar f)(g) \quad &= \quad \int_K f(x)\rho_\hbar(x^{-1}g)\,dx, \qquad f \in L^2(K, dx),\ g \in K_{\mathbb{C}}, \end{aligned} \tag{3.1}$$

where dx is the normalized Haar measure on K. For each $f \in L^2(K, dx)$, $C_\hbar f$ is the analytic continuation to $K_\mathbb{C}$ of the solution of the heat equation on K,

$$\frac{\partial u}{\partial \hbar} = \frac{1}{2}\Delta u,$$

with initial condition given by $u(0,x) = f(x)$. Therefore, $C_\hbar f$ is given by

$$(C_\hbar f)(g) = (\mathcal{C} \circ \rho_\hbar \star f)(g) = \left(\mathcal{C} \circ e^{\frac{\hbar\Delta}{2}} f\right)(g),$$

where $\star$ denotes the convolution in K and $\mathcal{C}$ denotes analytic continuation from K to $K_\mathbb{C}$. Recall that the K-averaged heat kernel measure $d\nu_\hbar$ on $K_\mathbb{C}$ is given by (2.22). Hall proves the following:

Theorem 3 (Hall). *For each $\hbar > 0$, the mapping $C_\hbar$ defined in (3.1) is an unitary isomorphism from $L^2(K, dx)$ onto the Hilbert space $\mathcal{H}L^2(K_\mathbb{C}, d\nu_\hbar) := L^2(K_\mathbb{C}, d\nu_\hbar) \bigcap \mathcal{H}(K_\mathbb{C})$.*

This transformation defines a Hilbert vector bundle $\mathcal{H}^{\mathrm{H}}$ over the one-dimensional base $\mathbb{R}_+$ with global coordinate $\hbar$,

$$\begin{array}{rcl} \mathcal{H}^{\mathrm{H}} & \to & \mathbb{R}_+ \\ \mathcal{H}^{\mathrm{H}}_\hbar & = & \mathcal{H}L^2(K_\mathbb{C}, d\nu_\hbar). \end{array}$$

which we call the *CST bundle*. The Hilbert space structure on the fibers of $\mathcal{H}^{\mathrm{H}}$ is defined by

$$\langle F_\hbar, G_\hbar\rangle^{\mathrm{H}}_\hbar := \int_{K_\mathbb{C}} \overline{F_\hbar(g)}\, G_\hbar(g)\, d\nu_\hbar(g), \quad F_\hbar, G_h \in \mathcal{H}^{\mathrm{H}}_\hbar.$$

From Theorem 3 we conclude that the operators

$$U^{\mathrm{H}}_{\hbar_2\hbar_1} = C_{\hbar_2} \circ C^{-1}_{\hbar_1} : \mathcal{H}^{\mathrm{H}}_{\hbar_1} \to \mathcal{H}^{\mathrm{H}}_{\hbar_2} \tag{3.2}$$

define unitary transformations between the fibers which satisfy

$$U^{\mathrm{H}}_{\hbar_3\hbar_2} \circ U^{\mathrm{H}}_{\hbar_2\hbar_1} = U^{\mathrm{H}}_{\hbar_3\hbar_1}.$$

These operators correspond to the parallel transport with respect to an Hermitian connection δ^{H} on $\mathcal{H}^{\mathrm{H}}$. Let $\hat{K}$ denote the set of (equivalence class of) irreducible unitary representations of K. It follows from Theorem 3 that by

choosing an orthonormal basis in the vector spaces V_R of all the representations $R \in \hat{K}$ the matrix entries $R_{ij}(\cdot)$ analytically continued to $K_{\mathbb{C}}$ form an orthogonal basis of $\mathcal{H}^{\mathrm{H}}_{\hbar}$

$$\{R_{ij}(\cdot)\}^{R\in\hat{K}}_{i,j=1,\dots,\ d_R} \subset \mathcal{H}^{\mathrm{H}}_{\hbar}$$

for all $\hbar > 0$, where d_R is the dimension of R, and

$$\int_{K_{\mathbb{C}}} \overline{R_{ij}(g)}\, R_{i'j'}(g)\, d\nu_\hbar(g) = \frac{e^{\hbar c_R}}{d_R}\ \delta_{ii'}\delta_{jj'}$$

where c_R is the eigenvalue of $-\Delta$ on the representation R. Therefore the sections of $\mathcal{H}^{\mathrm{H}}$

$$R_{ij}(\cdot)$$

form a global orthogonal frame and obviously the sections

$$F^{R_{ij}}_{\hbar} = e^{-\hbar c_R/2}\sqrt{d_R}\, R_{ij}(\cdot) \tag{3.3}$$

form a global orthonormal frame, that is, their norms do not depend on $\hbar$. Moreover

$$U^{\mathrm{H}}_{\hbar_2\hbar_1}(R_{ij}) = e^{-(\hbar_2-\hbar_1)c_R/2}\, R_{ij} \ \Leftrightarrow\ U^{\mathrm{H}}_{\hbar_2\hbar_1}(F^{R_{ij}}_{\hbar_1}) = F^{R_{ij}}_{\hbar_2}\ . \tag{3.4}$$

Definition 5. The *CST connection* δ^{H} on $\mathcal{H}^{\mathrm{H}}$ is the (unique) connection for which the sections (3.3) are horizontal

$$\delta^{\mathrm{H}}(F^{R_{ij}}) = 0. \tag{3.5}$$

The CST connection is automatically Hermitian. From (3.3), δ^{H} can be equivalently defined through

$$\delta^{\mathrm{H}}_{\frac{\partial}{\partial\hbar}}(R_{ij}) = \frac{c_R}{2}\, R_{ij} = -\frac{\Delta_{\mathbb{C}}}{4}\, R_{ij}.$$

Proposition 4. *The unitary parallel transport U corresponding to the connection (3.5) is given by the unitary operators (3.2), $U = U^{\mathrm{H}}$.*

Proof. Since the sections $F^{R_{ij}}$ satisfy (3.5) the parallel transport for them is

$$U_{\hbar_2\hbar_1}F^{R_{ij}}_{\hbar_1} = F^{R_{ij}}_{\hbar_2} \qquad \forall \hbar_1, \hbar_2 \in \mathbb{R}_+. \tag{3.6}$$

This map of orthonormal basis extends to a unique unitary isomorphism

$$U_{\hbar_2\hbar_1} : \mathcal{H}^{\mathrm{H}}_{\hbar_1} \to \mathcal{H}^{\mathrm{H}}_{\hbar_2}$$

From (3.4) and (3.6) we see that $U_{\hbar_2\hbar_1}$ and $U^{\mathrm{H}}_{\hbar_2\hbar_1}$ coincide on the basis vectors and therefore they coincide as unitary operators. $\square$

3.2 Equivalence between δ^{H} and δ^{Q}

Our main result in this section is

Theorem 4. *The quantum connection δ^{Q} and the CST connection δ^{H} are equivalent in the sense that there exists a natural unitary isomorphism $S : \mathcal{H}^{\mathrm{H}} \to \mathcal{H}^{\mathrm{Q}}$ such that*

$$\delta^{\mathrm{Q}}_{\frac{\partial}{\partial \hbar}} \circ S = S \circ \delta^{\mathrm{H}}_{\frac{\partial}{\partial \hbar}}. \tag{3.7}$$

Proof. We start by constructing a natural unitary isomorphism between the CST bundle $\mathcal{H}^{\mathrm{H}}$ and the quantum bundle $\mathcal{H}^{\mathrm{Q}}$. Let σ, ζ be two sections of $\mathcal{H}^{\mathrm{Q}}$ of the form (2.11), (2.12),

$$\begin{aligned} \sigma_s &= \hat{F}(xe^{isY})\, e^{-\frac{s|Y|^2}{2\hbar_0}} \sqrt{\Omega_s} \\ \zeta_s &= \hat{G}(xe^{isY})\, e^{-\frac{s|Y|^2}{2\hbar_0}} \sqrt{\Omega_s}. \end{aligned}$$

From proposition 2 we obtain

$$\langle \sigma_s, \zeta_s \rangle^Q = a_s \int_{K_{\mathbb{C}}} \overline{\hat{F}} \hat{G} d\nu_\hbar \ ,$$

where $\hbar = s\hbar_0$ and a_s was defined in (2.25). Therefore, the bundle morphism

$$\begin{aligned} S_\hbar \ : \ \mathcal{H}^{\mathrm{H}}_\hbar &\to \mathcal{H}^{\mathrm{Q}}_s \\ \hat{F} &\mapsto \psi_s^*(\hat{F})\, e^{-\frac{s|Y|^2}{2\hbar_0}} \sqrt{\tfrac{\Omega_s}{a_s}}, \end{aligned}$$

is a unitary isomorphism. To show (3.7) it is sufficient to see that the frame of horizontal sections

$$\left\{ F^{R_{ij}} = e^{-\hbar c_R/2} R_{ij} \right\}$$

is mapped to an horizontal frame. This follows directly from theorem 2. ∎

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